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Toda lattice field theories, discrete W algebras, Toda lattice hierarchies and quantum groups

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Abstract

In analogy with the Liouville case we study the sl_3 Toda theory on the lattice and define the relevant quadratic algebra and out of it we recover the discrete W_3 algebra. We define an integrable system with respect to the latter and establish the relation with the Toda lattice hierarchy. We compute the the relevant continuum limits. Finally we find the quantum version of the quadratic algebra.

1 Introduction

It is well–known that one can define and quantize the Liouville field theory on a lattice, [1],[2],[3],[4]. Following the last two references, it was shown in [5] that the same can be done with more general Toda field theories. Basically one exploits both the integrability properties of such theories, which carry over to the lattice in a straightforward way, and the conformal structure which reappears on the lattice in a discrete but recognizable way (see [6]). As for the quantum theory, one heavily relies on the quantum group representation theory.

In this paper we concentrate mostly on the discrete algebras that underlie the Toda lattice theories and on the relation with integrable hierarchies and matrix models. It is true that one can find analogous algebras and relations in a continuous parallel treatment of these models, but it is striking to find a kind of universality in a lattice approach (see for example the coincidence of our algebra (3.5) with that of [7]).

The paper is organized as follows. In section 2, of pedagogical character, we collect mostly well–known statements and results about the discrete Liouville model and its connection with the KdV hierarchy and the 1–matrix model. In section 3, starting from the SL(3) Toda field theory on the lattice, we compute the discrete W_3 algebra and construct an associated integrable model; the latter turns out to be a sort of linearization of the Toda lattice hierarchy. In section 4 we present the quadratic discrete quantum W_3 algebra. Section 5 is devoted to a discussion: we explain in particular in what sense the problems studied here may be connected to 2D gravity and string theory.

2 Liouville theory on the lattice and the KdV case.

In this section we essentially collect known results concerning the Liouville theory on the lattice and one–matrix model from refs.[3] and [9], respectively. In [3] it was shown that to each chiral half of a Liouville field theory we can associate a lattice field theory, which (in analogy with the continuum case, [8]) is characterized by a (canonical) quadratic algebra with generators $W_n^{(i)}$, $i = 1, 2$ and n is the lattice variable (discretized space). The generators are constructed according to

$$W_n^{(i)} = \sigma_n^1 \sigma_{n+i}^2 - \sigma_n^2 \sigma_{n+i}^1 \tag{2.1}$$

where σ_n^i are the discrete analogs of the two distinct solutions of the second order differential equation $(\partial^2 + u)\sigma = 0$, which underlies the Liouville theory – u is the chiral energy–momentum tensor. Then, the quadratic algebra is

$$\begin{aligned}
\left\{W_n^{(1)}\,,\,W_m^{(1)}\right\} &= W_n^{(1)} W_m^{(1)} (\delta_{n,m-1} - \delta_{n,m+1}) \\
\left\{W_n^{(1)}\,,\,W_m^{(2)}\right\} &= W_n^{(1)} W_m^{(2)} (-\delta_{n,m+2} + \delta_{n,m+1} - \delta_{n,m} + \delta_{n,m-1}) \\
\left\{W_n^{(2)}\,,\,W_m^{(2)}\right\} &= W_n^{(2)} W_m^{(2)} (\delta_{n,m-2} - 2\delta_{n,m-1} + 2\delta_{n,m+1} - \delta_{n,m+2}) + \\
&\quad - 4W_{n-1}^{(1)} W_{n+1}^{(1)} \delta_{n,m+1} + 4W_n^{(1)} W_{n+2}^{(1)} \delta_{n,m-1}
\end{aligned} \tag{2.2}$$

But if we introduce the quantity

$$S_n = 4\frac{W_{n+1}^{(1)} W_{n-1}^{(1)}}{W_n^{(2)} W_{n-1}^{(2)}} \tag{2.3}$$

the algebra (2.2) shows two simple factors: the subalgebra spanned by $W_n^{(1)}$ and

$$\begin{aligned} \{S_n\,,\,S_m\} &= S_nS_m\,[(4-S_n-S_m)(\delta_{n,m+1}-\delta_{n,m-1}) \\ &\quad +S_{n+1}\delta_{n,m-2}-S_{n-1}\delta_{n,m+2}] \end{aligned} \tag{2.4}$$

while

$$\left\{W_n^{(1)}\,,\,S_m\right\}=0 \tag{2.5}$$

$W_n^{(1)}$ carries antichiral degrees of freedom, which decouple from the theory in the continuum limit and are an artifact of the lattice regularization. Therefore the significant algebra is (2.4).

The continuum limit is obtained with the following substitutions

$$\begin{aligned} & i\to x, \qquad j\to y, \\ & S_i\to 1+\epsilon^2 u(x)+\ldots, \qquad S_{i+n}\to 1+\epsilon^2\left(\sum_{l=0}\frac{(n\epsilon)^l}{l!}u^{(l)}(x)\right) \\ & \delta_{i,j}\to\epsilon\delta(x,y), \qquad \delta_{i+n,j}\to\epsilon\left(\sum_{l=0}\frac{(n\epsilon)^l}{l!}\delta^{(l)}(x,y)\right) \end{aligned} \tag{2.6}$$

We find

$$\lim_{\epsilon\to 0}\frac{1}{4\epsilon^4}RHS=\{u(x),u(y)\}_2 \tag{2.7}$$

RHS means the right hand side of (2.4) after the substitution, and

$$\{u(x),u(y)\}_2=(2u(x)\partial_x+u'(x)+\frac{1}{2}\partial_x^3)\delta(x,y) \tag{2.8}$$

is the well–known second KdV Poisson structure.

It is remarkable that the algebra (2.4) is met in one–matrix models with even potential, [9]. In this case the semiinfinite matrix Q, which represents the multiplication by the eigenvalue in the orthogonal polynomial approach, takes the following form:

$$Q=\sum_{i\geq 0}\Big(E_{i,i+1}+R_iE_{i,i-1}\Big), \qquad (E_{ij})_{k,l}=\delta_{i,k}\delta_{j,l}$$

One finds two compatible Poisson structures:

$$\{R_i,R_j\}_1 = R_iR_{i+1}\delta_{j,i+1}-R_iR_{i-1}\delta_{j,i-1} \tag{2.9}$$

$$\begin{aligned} \{R_i,R_j\}_3 &= R_iR_{i+1}(R_i+R_{i+1})\delta_{j,i+1}-R_iR_{i-1}(R_i+R_{i-1})\delta_{j,i-1} \\ &\quad R_iR_{i+1}R_{i+2}\delta_{j,i+2}-R_iR_{i-1}R_{i-2}\delta_{j,i-2} \end{aligned} \tag{2.10}$$

These are the two homogeneous components of the algebra (2.4). More precisely we have

$$\{\cdot,\cdot\}=\{\cdot,\cdot\}_3-4\{\cdot,\cdot\}_1$$

One may wonder whether it is possible to define an infinite set of conserved quantities and integrable flow equations in correspondence with the algebra of the S_n's or R_n's, which

we formally identify in the following. The answer is easily provided by the matrix model. The hamiltonians are defined as $H_{2k} = trQ^{2k}$, so that

$$H_2 = \sum_n R_n, \qquad H_4 = \sum_n (\frac{1}{2}R_n^2 + R_n R_{n+1}), \ldots$$

and the flows are given by

$$\frac{\partial Q}{\partial \tau_k} = [Q_+^k, Q]_- \tag{2.11}$$

where the – subscript means that we are considering only the strictly lower triangular part of the semiinfinite matrix, while + denotes the upper triangular part (including the main diagonal). In particular

$$\begin{aligned}
\frac{\partial R_n}{\partial t_2} &= R_n(R_{n+1} - R_{n-1}) \equiv \{R_n, H_2\}_1 \\
\frac{\partial R_n}{\partial t_4} &= R_n R_{n+1}\Big(R_n + R_{n+1} + R_{n+2}\Big) - R_n R_{n-1}\Big(R_n + R_{n-1} + R_{n-2}\Big) \equiv \\
&\{R_n, H_4\}_1 = \{R_n, H_2\}_3
\end{aligned} \tag{2.12}$$

In the continuum limit we find

$$\begin{aligned}
&\lim_{\epsilon\to 0}\Big(\frac{1}{2\epsilon^3}\frac{\partial R_n}{\partial t_2}\Big) = u' \equiv \frac{\partial u}{\partial \tilde{t}_1} \\
&\lim_{\epsilon\to 0}\Big[\frac{2}{\epsilon^5}\Big(-\frac{\partial R_n}{\partial t_2} + \frac{1}{4}\frac{\partial R_n}{\partial t_4}\Big)\Big] = u''' + 6uu' \equiv \frac{\partial u}{\partial \tilde{t}_3}
\end{aligned}$$

and so on. The RHS's define the renormalized couplings $\tilde{t}_1, \tilde{t}_3, \ldots$.

Therefore we can call eqs.(2.12) discrete KdV flows.

3 Discrete W_3 and discrete Boussinesq

In ref.[5] it was shown that discretizing the SL(3) Toda theory on the lattice produces (canonically) a quadratic algebra with generators $W_n^{(i)}$, $i = 1, 2, 3$, defined as follows

$$W_n^{(i)} = (-1)^i \varepsilon^{ijkl} \sigma^1_{n+j} \sigma^2_{n+k} \sigma^3_{n+l}, \qquad i, j, k, l = 0, 1, 2, 3 \tag{3.1}$$

where ε^{ijkl} is the standard constant completely antisymmetric tensor; it follows, in particular, that $W_n^{(0)} = W_{n+1}^{(3)}$. The σ_n^i are the discrete analogs of the three distinct solutions of the equation

$$(\partial^3 + u\partial + w)\sigma = 0 \tag{3.2}$$

where u, w are constructed out of the Toda fields. The quadratic algebra is

$$\begin{aligned}
\Big\{W_n^{(1)}, W_m^{(1)}\Big\} &= -\frac{1}{3}W_n^{(1)}W_m^{(1)}(\delta_{n,m-3} - \delta_{n,m-1} + \delta_{n,m+1} - \delta_{n,m+3}) + \\
&\quad + W_n^{(2)}W_{n+2}^{(3)}\delta_{n,m-1} - W_{n+1}^{(3)}W_{n-1}^{(2)}\delta_{n,m+1} \\
\Big\{W_n^{(1)}, W_m^{(2)}\Big\} &= -\frac{1}{3}W_n^{(1)}W_m^{(2)}(\delta_{n,m-3} + \delta_{n,m-2} - 2\delta_{n,m-1} +
\end{aligned}$$

$$
\begin{aligned}
& \qquad + \delta_{n,m} - \delta_{n,m+1} + \delta_{n,m+2} - \delta_{n,m+3}) + \qquad (3.3)\\
& \qquad + W^{(3)}_n W^{(3)}_{n+2}\delta_{n,m-1} - W^{(3)}_{n+1}W^{(3)}_{n-2}\delta_{n,m+2}\\
\left\{W^{(1)}_n\,,\,W^{(3)}_m\right\} &= -\frac{1}{3}W^{(1)}_n W^{(3)}_m(\delta_{n,m-3} + \delta_{n,m-2} - \delta_{n,m-1} - \delta_{n,m+2})\\
\left\{W^{(2)}_n\,,\,W^{(2)}_m\right\} &= -\frac{1}{3}W^{(2)}_n W^{(2)}_m(\delta_{n,m-3} - \delta_{n,m-1} + \delta_{n,m+1} - \delta_{n,m+3}) +\\
& \qquad - W^{(3)}_n W^{(1)}_{n+1}\delta_{n,m-1} + W^{(1)}_n W^{(3)}_{n-1}\delta_{n,m+1}\\
\left\{W^{(2)}_n\,,\,W^{(3)}_m\right\} &= -\frac{1}{3}W^{(2)}_n W^{(3)}_m(\delta_{n,m-3} + \delta_{n,m} - \delta_{n,m+1} - \delta_{n,m+2})\\
\left\{W^{(3)}_n\,,\,W^{(3)}_m\right\} &= -\frac{1}{3}W^{(3)}_n W^{(3)}_m(\delta_{n,m-2} + \delta_{n,m-1} - \delta_{n,m+1} - \delta_{n,m+2})
\end{aligned}
$$

As in the previous section this algebra contains two irreducible factors. One is the subalgebra generated by $W^{(3)}_n$, the other is generated by S_n and W_n, which are defined as follows:

$$
S_n = -\frac{W^{(2)}_{n-1}W^{(3)}_{n+1}}{W^{(1)}_{n-1}W^{(1)}_n}, \qquad W_n = Z_n S_n S_{n+1}, \qquad Z_n = \frac{W^{(1)}_n W^{(3)}_{n-1}}{W^{(2)}_{n-1}W^{(2)}_n} \tag{3.4}
$$

It is lengthy but straightforward to prove that *

$$
\{W^{(3)}_n, S_m\} = 0, \qquad\qquad \{W^{(3)}_n, W_m\} = 0
$$

and

$$
\begin{aligned}
\{S_n\,,\,S_{n+1}\} &= (S_n S_{n+1} - W_n)\,(1 - S_n - S_{n+1})\\
\{S_n\,,\,S_{n+2}\} &= -S_n S_{n+1} S_{n+2} + W_n S_{n+2} + W_{n+1} S_n
\end{aligned}
$$

$$
\begin{aligned}
\{W_n\,,\,S_{n-1}\} &= -W_n S_{n-1}\,(1 - S_n - S_{n-1}) - W_n W_{n-1}\\
\{W_n\,,\,S_{n-2}\} &= W_n S_{n-1} S_{n-2} - W_n W_{n-2}\\
\{W_n\,,\,S_n\} = -\{W_n\,,\,S_{n+1}\} &= W_n S_n S_{n+1} - W_n^2\\
\{W_n\,,\,S_{n+2}\} &= W_n S_{n+2}\,(1 - S_{n+1} - S_{n+2}) + W_n W_{n+1}\\
\{W_n\,,\,S_{n+3}\} &= -W_n S_{n+2} S_{n+3} + W_n W_{n+2}
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
\{W_n\,,\,W_{n+1}\} &= W_n W_{n+1}\,(1 - S_n - S_{n+2})\\
\{W_n\,,\,W_{n+2}\} &= W_n W_{n+2}\,(1 - S_{n+1} - S_{n+2})\\
\{W_n\,,\,W_{n+3}\} &= -W_n W_{n+3} S_{n+2}
\end{aligned}
$$

The other brackets are either obtained in an obvious way from these or else vanish.

Motivated by what follows we call this the discrete W_3 algebra. It is striking that this algebra was found some time ago by A.A.Belov and K.D.Chaltikian [7], with a different approach based on Feigin's construction of lattice screening charges. It is common wisdom that there may by many ways to discretize a continuous system. This example seems to imply however that we can find universal structures also in lattice theories. This is due to the fact that conformal and W structures persist in a discrete form in lattice theories.

*This verification has been first done by V.Bonservizi.

3.1 The continuum limit

The continuum limit of the discrete W_3 algebra is expected to lead to the well-known classical continuous W_3 algebra. The technical aspects of such limit do not seem to have been carefully carried out up to now and are not as straightforward as in the previous section. Therefore we spend a few words on them.

We start from the expansions

$$S_n \to \frac{1}{3} + \frac{1}{9}\epsilon^2 u(x) + \dots, \qquad W_n \to \frac{1}{27}\Big(1 + \epsilon^2 u(x) + \epsilon^3 w(x)\Big) + \dots \tag{3.6}$$

They are expected to lead to the following classical W_3 algebra

$$\{u(x), u(y)\} = (2u(x)\partial_x + u'(x) + 2\partial_x^3)\delta(x,y) \tag{3.7a}$$
$$\{u(x), w(y)\} = (3w(x)\partial_x + 2w'(x) - \partial_x^2 u(x) - \partial_x^4)\delta(x,y) \tag{3.7b}$$
$$\{w(x), w(y)\} = \Big(2w'(x)\partial_x + w''(x) - \frac{2}{3}(u(x) + \partial_x^2)(\partial_x u(x) + \partial_x^3)\Big)\delta(x,y) \tag{3.7c}$$

Then we write the algebra (3.5) in terms of Kronecker δ symbols (like in (2.4)) and replace the RHS's with the relative continuous expressions. We denote by $CL(X,Y)$ the RHS of $\{X_n\,,\, Y_m\}$ after this substitution, where X_n and Y_n stand for either S_n or W_n. We expect to retrieve the continuous algebra by finding suitable combinations of the $CL(X,Y)$ so that when $\epsilon \to 0$ one gets one of the commutators (3.7a–3.7c).

In fact we find

$$\lim_{\epsilon\to 0}\frac{3^4}{\epsilon^4}\Big(-\,CL(S,S)\Big) = \{u(x), u(y)\}$$
$$\lim_{\epsilon\to 0}\frac{3^5}{\epsilon^5}\Big(CL(W,S) - 3\,CL(W,W)\Big) = \{u(x), w(y)\}$$
$$\lim_{\epsilon\to 0}\frac{3^4}{\epsilon^6}\Big(-\,CL(S,S) + 3CL(S,W) + 3CL(W,S) - 9CL(W,W)\Big) = \{w(x), w(y)\}$$

It is evident that one can get the same result directly from (3.3).

3.2 An integrable system

As in the previous section we may wonder whether we can associate an integrable system to the Poisson structures (3.5). We use the plural because, in fact, (3.5) contains four homogeneous algebras: if we assign to S_n and W_n degree 2 and 3, respectively, we see that the right hand sides of (3.5) contains terms of degree 3,4,5,6. Accordingly we have four Poisson brackets, which we explicitly write down for convenience

$$\{S_n\,,\, S_{n+1}\}_0 \;=\; -W_n \tag{3.8}$$

$$\begin{aligned}
&\{S_n\,,\, S_{n+1}\}_1 \;=\; S_n S_{n+1}, \\
&\{W_n\,,\, S_{n-1}\}_1 \;=\; -W_n S_{n-1}, \qquad \{W_n\,,\, S_{n+2}\}_1 \;=\; W_n S_{n+2} \\
&\{W_n\,,\, W_{n+1}\}_1 \;=\; W_n W_{n+1}, \qquad \{W_n\,,\, W_{n+2}\}_1 \;=\; W_n W_{n+2}
\end{aligned} \tag{3.9}$$

$$
\begin{aligned}
&\{S_n\,,\,S_{n+1}\}_2 \,=\, W_n(S_n+S_{n+1}), \qquad \{S_n\,,\,S_{n+2}\}_2 \,=\, W_nS_{n+2}+W_{n+1}S_n\\
&\{W_n\,,\,S_{n-2}\}_2 \,=\, -W_nW_{n-2}, \qquad \{W_n\,,\,S_{n-1}\}_2 \,=\, -W_nW_{n-1}\\
&\{W_n\,,\,S_n\}_2 \,=\, -W_n^2, \qquad \{W_n\,,\,S_{n+1}\}_2 \,=\, W_n^2\\
&\{W_n\,,\,S_{n+2}\}_2 \,=\, W_nW_{n+1}, \qquad \{W_n\,,\,S_{n+3}\}_2 \,=\, W_nW_{n+2}
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
&\{S_n\,,\,S_{n+1}\}_3 \,=\, -S_nS_{n+1}(S_n+S_{n+1}), \qquad \{S_n\,,\,S_{n+2}\}_3 \,=\, -S_nS_{n+1}S_{n+2}\\
&\{W_n\,,\,S_{n-2}\}_3 \,=\, W_nS_{n-1}S_{n-2}, \qquad \{W_n\,,\,S_{n-1}\}_3 \,=\, W_nS_{n-1}(S_n+S_{n-1})\\
&\{W_n\,,\,S_n\}_3 \,=\, W_nS_nS_{n+1}, \qquad \{W_n\,,\,S_{n+1}\}_3 \,=\, -W_nS_nS_{n+1}\\
&\{W_n\,,\,S_{n+2}\}_3 \,=\, -W_nS_{n+2}(S_{n+1}+S_{n+2}), \qquad \{W_n\,,\,S_{n+3}\}_3 \,=\, -W_nS_{n+2}S_{n+3}\\
&\{W_n\,,\,W_{n+1}\}_3 \,=\, -W_nW_{n+1}(S_n+S_{n+2}),\\
&\{W_n\,,\,W_{n+2}\}_3 \,=\, -W_nW_{n+2}(S_{n+1}+S_{n+2}), \qquad \{W_n\,,\,W_{n+3}\}_3 \,=\, -W_nW_{n+3}S_{n+2}
\end{aligned} \tag{3.11}
$$

All the other brackets can either be obtained from the above in an obvious way or else vanish.

One can define new (inhomogeneous) Poisson structures as follows

$$
\{\cdot,\cdot\}^{(1)} = \{\cdot,\cdot\}_0 + \{\cdot,\cdot\}_1, \qquad \{\cdot,\cdot\}^{(2)} = \{\cdot,\cdot\}_2 + \{\cdot,\cdot\}_3
$$

With respect to the new Poisson brackets it is possible to define a bi–hamiltonian system [7]. The first three hamiltonians were introduced in [7]:

$$
\begin{aligned}
\mathcal{H}_1 &= \sum_n S_n\\
\mathcal{H}_2 &= \sum_n \Big(W_n - \frac{1}{2}S_n^2 - S_nS_{n+1}\Big)\\
\mathcal{H}_3 &= \sum_n \Big(\frac{1}{3}S_n^3 + S_nS_{n+1}(S_n+S_{n+1}+S_{n+2}) - W_n(S_{n-1}+S_n+S_{n+1}+S_{n+2})\Big)
\end{aligned} \tag{3.12}
$$

We will shortly see how to generalize these first few cases with a very simple construction. The two Poisson brackets defined above are compatible with respect to these hamiltonians *provided we disregard all the third power terms in* W_n. Then, the flows are defined by

$$
\frac{\partial f_n}{\partial t_i} \,=\, \{f_n\,,\,\mathcal{H}_i\}^{(2)} = \{f_n\,,\,\mathcal{H}_{i+1}\}^{(1)} \tag{3.13}
$$

where f_n is either W_n or S_n. They will contain terms at most quadratic in W_n due to the above (essential) truncation. We call the model characterized by these truncated flows the BC model, [7]. From these discrete flows one can extract the continuous Boussinesq flows with the procedure of the previous subsection.

$$
\lim_{\epsilon\to 0}\frac{3^3}{\epsilon^3}\,\frac{\partial S_n}{\partial t_2} = u'(x) \equiv \frac{\partial u}{\partial \tilde{t_1}}, \qquad \lim_{\epsilon\to 0}\frac{3^3}{\epsilon^4}\Big(3\,\frac{\partial W_n}{\partial t_2} - \frac{\partial S_n}{\partial t_2}\Big) = w'(x) \equiv \frac{\partial w}{\partial \tilde{t_1}}
$$

and

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\frac{3^3}{\epsilon^4}\Big(4\frac{\partial S_n}{\partial t_2} - 3\frac{\partial S_n}{\partial t_3}\Big) = 2w'(x) - u''(x) \equiv \frac{\partial u}{\partial \tilde{t_2}}\\
&\lim_{\epsilon\to 0}\frac{3^3}{\epsilon^5}\Big(-4\frac{\partial S_n}{\partial t_2} + 12\frac{\partial W_n}{\partial t_2} + 3\frac{\partial S_n}{\partial t_3} - 9\frac{\partial W_n}{\partial t_3}\Big) = w''(x) - \frac{2}{3}(u(x)u'(x) + u'''(x)) \equiv \frac{\partial w}{\partial \tilde{t_2}}
\end{aligned}
$$

and so on. The rightmost derivatives define the continuous flow parameters.

3.3 Connection with the Toda lattice hierarchy

In this subsection we intend to study the relationship between the truncated BC model and the Toda lattice hierarchy. We shall see that the BC model can be reconstructed from the Toda lattice hierarchy provided we consider a suitable linearization of the latter. This will automatically establish a relation with two–matrix models, see [10].

The connection between Toda lattice hierarchy and differential N–KdV hierarchies was studied in [10]. By Toda lattice hierarchy we simply mean here the integrable flows

$$\frac{\partial Q}{\partial \tau_k} = [Q_+^k, Q] \tag{3.14}$$

where Q is a semi–infinite matrix.

In particular, as far as the 3–KdV or Boussinesq hierarchy is concerned, it has been shown that the relevant Q matrix has the form

$$\bar{Q} = \sum_{j=0}^{\infty} \Big(E_{j,j+1} + a_j E_{j+l,j} + b_j E_{j+2,j} \Big), \qquad (E_{j,m})_{k,l} = \delta_{j,k}\delta_{m,l} \tag{3.15}$$

For later convenience we represent the matrix $\bar{Q}$ by the following operator

$$\bar{Q}(j) = e^{\partial_0} + a_j e^{-\partial_0} + b_j e^{-2\partial_0} \tag{3.16}$$

where e^{∂_0} is the finite shift operator defined by $(e^{\partial_0} f)_j = f_{j+1}$, for any discrete function f_j.

The contact between (3.16) and (3.15) is made by acting with the former on a discrete function $\xi(j)$; then $\bar{Q}(j)\xi(j)$ is the same as the j–th component of $\bar{Q}\xi$, where ξ is a column vector with components $\xi(0), \xi(1), \ldots$. We will generally understand the dependence on j in (3.16).

If we now define

$$\mathrm{H}_k = \mathrm{Tr}(\bar{Q}^k) = \sum_j \bar{Q}^k_{(0)}(j)$$

where the subscript $_{(p)}$ denotes the coefficient of $e^{-p\partial_0}$ in $\bar{Q}^k(j)$, and make the identifications

$$a_n \equiv S_n, \qquad\qquad b_n \equiv W_{n-1} \tag{3.17}$$

we find a remarkable recursion relation

$$\{f_n\,,\,\mathrm{H}_k\}_0 - \{f_n\,,\,\mathrm{H}_{k-1}\}_1 + \{f_n\,,\,\mathrm{H}_{k-2}\}_2 - \{f_n\,,\,\mathrm{H}_{k-3}\}_3 = 0 \tag{3.18}$$

which holds for any k, provided we set $\mathrm{H}_k = 0$, $k \le 1$.

However the matrix $\bar{Q}$ cannot be just plugged in eq.(3.14), because this would lead to inconsistencies. In fact, in related treatments, Eq.(3.15) is only a starting point, but, in order to get the 3–KdV flows, one has either to go through a process of hamiltonian reduction or to introduce suitable corrections to (3.15), as in [11]. Since we want to find a relation between the BC model and the Toda lattice hierarchy, it is reasonable to start from (3.15) and (3.14), but one expects to have to introduce suitable modifications.

Let us start from the *linearized* hamiltonians

$$H_k = \mathrm{Tr}(\bar{Q}^k_\ell)$$

where the subscript ℓ means that in $\bar{Q}^k$ we keep only terms which are independent of or linear in the field b. Then, if we find that the BC hamiltonians are reproduced by

$$\mathcal{H}_k = (-1)^{k+1}(H_{2k} - H_{2k-1}), \qquad k = 1, 2, ... \qquad H_1 = 0 \tag{3.19}$$

This formula, with the identification (3.17), provides a generalization of (3.12).

The derivation of the BC flows is more elaborate. First of all let us introduce the operators $A(j) = a_j e^{-\partial_0}$ and $B(j) = b_j e^{-\partial_0}$. Then we write down the linearized flows

$$\frac{\partial \bar{Q}}{\partial \tau_k} = [(\bar{Q}^k_\ell)_+, \bar{Q}]_- + [\Big(A\bar{Q}^{k-1}_\ell\Big)_{(0)}, B]e^{-\partial_0} \tag{3.20}$$

Next we define

$$\frac{\partial a_n}{\partial \tau_k} \equiv \Big(\frac{\partial a_n}{\partial \tau_k}\Big)_a + \Big(\frac{\partial a_n}{\partial \tau_k}\Big)_b \tag{3.21}$$

i.e. we split the a flows into a b–independent and a b–dependent part. As a consequence of the above definitions it turns out that the b–dependent part of the a flows vanish for odd k. The rightmost correction term in (3.20) affects only the b flows.

Finally, after the substitution (3.17), we find that the BC flows can be expressed in terms of the corrected Toda flows (3.20) as follows

$$\begin{aligned} \frac{\partial S_n}{\partial t_k} &= \Big(\frac{\partial S_n}{\partial \tau_{2k}}\Big)_a - \frac{\partial S_n}{\partial \tau_{2k-1}} + \Big(\frac{\partial S_n}{\partial \tau_{2k-2}}\Big)_b \\ \frac{\partial W_n}{\partial t_k} &= \frac{\partial W_n}{\partial \tau_{2n}} - \frac{\partial W_n}{\partial \tau_{2k-1}} \end{aligned} \tag{3.22}$$

4 Quantum discrete W algebra

Using the quantum exchange relations of ref.[5], we compute the quantum version of (3.5). First, inspired by the definition of quantum determinant in $\mathrm{SL_q}(3)$, we define the quantum $W_n^{(i)}$ generators as follows

$$W_n^{(i)} = (-1)^i \tilde{\varepsilon}^{ijkl} \sigma^1_{n+j} \sigma^2_{n+k} \sigma^3_{n+l}, \qquad i, j, k, l = 0, 1, 2, 3 \tag{4.23}$$

where the ordering of the σ^i_n is now crucial and

$$\tilde{\varepsilon}^{ijkl} = \varepsilon^{ijkl} q^p, \qquad q = e^{-i\hbar}$$

where p is the number of contiguous exchanges to pass from the ordering jkl to the natural ordering of the same integers. Then we compute

$$\begin{aligned} W_n^{(1)}\,\sigma_m &= \sigma_m\, W_n^{(1)}\, q^{-\frac{2}{3}(\delta_{m,n} - \delta_{m,n+1} - \delta_{m,n+3})} + A\,\sigma_n\, W_m^{(3)} \delta_{m,n+1} \\ W_n^{(2)}\,\sigma_m &= \sigma_m\, W_n^{(2)}\, q^{-\frac{2}{3}(\delta_{m,n+2} + \delta_{m,n} - \delta_{m,n+3})} - B\,\sigma_{n+3}\, W_n^{(3)} \delta_{m,n+2} \\ W_n^{(3)}\,\sigma_m &= \sigma_m\, W_n^{(3)}\, q^{\frac{2}{3}(\delta_{m,n+2} - \delta_{m,n})} \end{aligned} \tag{4.24}$$

where

$$A = q^{-\frac{2}{3}} - q^{\frac{4}{3}}, \qquad B = q^{\frac{2}{3}} - q^{-\frac{4}{3}}$$

The exchange relations (4.24) lead to

$$
\begin{aligned}
W_n^{(1)}\,W_m^{(1)} &= W_m^{(1)}\,W_n^{(1)}\,q^{\frac{2}{3}(\delta_{m,n+3}-\delta_{m,n+1}+\delta_{m,n-1}-\delta_{m,n-3})}\\
&\quad -A\,W_{n-1}^{(2)}\,W_{n+1}^{(3)}\delta_{m,n-1}-\,B\,W_n^{(2)}\,W_{n+2}^{(3)}\delta_{m,n+1}\\
W_n^{(1)}\,W_m^{(2)} &= W_m^{(2)}\,W_n^{(1)}\,q^{\frac{2}{3}(\delta_{m,n+3}+\delta_{m,n+2}-2\delta_{m,n+1}+\delta_{m,n}-\delta_{m,n-1}+\delta_{m,n-2}-\delta_{m,n-3})}\\
&\quad -B\,W_{n+2}^{(3)}\,W_n^{(3)}\delta_{m,n+1}-\,A\,W_{n-2}^{(3)}\,W_{n+1}^{(3)}\delta_{m,n-2}\\
W_n^{(1)}\,W_m^{(3)} &= W_m^{(3)}\,W_n^{(1)}\,q^{\frac{2}{3}(\delta_{m,n+3}+\delta_{m,n+2}-\delta_{m,n+1}-\delta_{m,n-2})}\\
W_n^{(2)}\,W_m^{(2)} &= W_m^{(2)}\,W_n^{(2)}\,q^{\frac{2}{3}(\delta_{m,n+3}-\delta_{m,n+1}+\delta_{m,n-1}-\delta_{m,n-3})}\\
&\quad +B\,W_n^{(3)}\,W_{n+1}^{(1)}\delta_{m,n+1}+\,A\,W_{n-1}^{(3)}\,W_n^{(1)}\delta_{m,n-1}\\
W_n^{(2)}\,W_m^{(3)} &= W_m^{(3)}\,W_n^{(2)}\,q^{\frac{2}{3}(\delta_{m,n+3}+\delta_{m,n}-\delta_{m,n-1}-\delta_{m,n-2})}\\
W_n^{(3)}\,W_m^{(3)} &= W_m^{(3)}\,W_n^{(3)}\,q^{\frac{2}{3}(\delta_{m,n+2}+\delta_{m,n+1}-\delta_{m,n-1}-\delta_{m,n-2})}
\end{aligned}
\tag{4.25}
$$

We can recover the classical limit (3.3) by expanding in $\hbar$ and setting $\hbar = \frac{1}{4}$.

We notice that the expressions (3.4) for S_n, W_n are ambiguous in the quantum case. One should suitably order them. However, no matter how one orders them, $W_n^{(3)}$ quantum commutes with them and therefore presumably constitutes a simple factor of the entire quantum algebra, as in the classical case.

5 Discussion

The results we have obtained above can no doubt be extended to sl_n Toda field theories: in the general case we would obtain a quadratic algebra with n generators and we would end up with the n–KdV hierarchy. Let us also recall that what we have done in this paper can be done in the continuum. Compared to the continuum treatment the lattice formalism may look awkward but offers some advantages: it provides an automatically regularized theory, it allows us to deal with quantum vertex operators (the σ_n^i's) in a very elegant way – in general the operator content of the quantum theory is better expressed on the lattice – and it provides a more direct connection with integrable hierarchies and matrix models. In brief continuum and lattice formalisms are usefully complementary to each other.

Anyhow, both the continuum and the discrete formalism reveal a triangular relation among Toda field theories, n–KdV hierarchies of differential equations and matrix models. Now let us also recall that the n-KdV hierarchies underlie the A_{n-1} topological field theory – in the sense that the correlation functions of the latter are determined by the flow equations of the former [12]. In turn these theories are the twisted versions of N=2 supersymmetric Landau–Ginzburg models. Therefore we see here *a correspondence between Toda field theories and N=2 supersymmetric models*. Indications of a contiguity between Toda field theories and N=2 superconformal models have been found also in [16]. In turn N=2 superconformal field theories brings into the game string vacua. On the other hand topological field theories, such as those mentioned above, can be obtained directly from matrix models, which seem to be natural carriers of topological degrees of freedom. We are thus led to a web of relationships. Although rather speculative at this stage, we do not think they are an accidental coincidence. We instead believe that there is a deep relation among the various facets mentioned above. Unfortunately, while each aspect separately has been considerably studied, a synoptic analysis is still incomplete. For example, it has been shown

in [13],[14], that n–KdV hierarchies describe topological properties of the moduli space of Riemann surfaces. This immediately brings to one's mind the role of the Liouville equation in the uniformization theory of Riemann surfaces. A comparison between KdV and the Liouville theory in this context is actually tried in [15]. But this is the only example, to our knowledge. The same can be said about the relations among Toda field theories, matrix models and N=2 superconformal field theories. But we would not be surprised to eventually find out that all these subjects are different facets of the same theory. This line of thinking was the original motivation of our research.

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